Discrete Mathematics 2  
Test File  
Fall 2009  

Exam #1  

1.) Prove the following identity in a Boolean algebra, justifying each step by quoting one of the properties of a Boolean algebra.

\[(a + b)(a' c)' = a + bc'\]

2.) Write a sum-of-products Boolean algebra expression for the following truth table.

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<th>a</th>
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3.) For the following pair of circuits, determine whether or not they are equivalent. If they are not equivalent, give an input that demonstrates they are not.

\[ab + ab' \text{ and } a\]

4.) How many positive integers less than or equal to 3000 are divisible by 5 or 7?

5.) What power of 5 divides 123!?

6.) Explain why 100! has two more digits than 99!.

7.) Find all cycles for the function \(f: \mathbb{N} \to \mathbb{N}\) defined by

\[g(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{n}{3} & \text{if } n \text{ is odd and divisible by 3} \\
 n + 1 & \text{otherwise}
\end{cases}\]

Prove your answer is correct by mathematical induction.

8.) Fill in the blanks to make a true statement out of the following. Explain how you got your answer.

\[0.25n^2 - 10n + 3 \in \Theta(n^2) \text{ because}\]
\[
\,
\]
for all \( n \geq \).

9.) Consider \( a_n \), the number of \( n \) digit even numbers.
   a.) Find a recursive model for \( a_n \).
   b.) Find a closed form formula for \( a_n \).

10.) Using sequences of differences, find a closed formula for the following sequence.

\[
1, 10, 35, 84, 165, 286, \ldots
\]

Practice Test

1.) Prove the following identity in a Boolean algebra, justifying each step by quoting one of the properties of a Boolean algebra.

\[
(a + b)(b + c) = ac + b
\]

2.) Write a sum-of-products Boolean algebra expression for the following truth table.

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3.) For the following pair of circuits, determine whether or not they are equivalent. If they are not equivalent, give an input that demonstrates they are not.

\[a'b + ab' \text{ and } ab\]

4.) How many positive integers less than or equal to 2000 are divisible by 3 or 5?

5.) How many consecutive zeros does the number 58! have on its right end?

6.) What power of 6 divides 78!?

7.) Find all cycles for the function \( f : \mathbb{N} \rightarrow \mathbb{N} \) defined by

\[
g(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
n + 5 & \text{if } n \text{ is odd}
\end{cases}
\]

Prove your answer is correct by mathematical induction.

8.) Fill in the blanks to make a true statement out of the following. Explain how you got your answer.
\[ 2n+1 \in \Theta(n) \text{ because } n \leq 2n+1 \leq 2n \text{ for all } n \geq 1. \]

9.) Consider \( a_n \), the number of \( n \) digit numbers that do not use the digit 0.
   a.) Find a recursive model for \( a_n \).
   b.) Find a closed form formula for \( a_n \).

10.) Using sequences of differences, find a closed formula for the following sequence.
    \[ 1, 3, 8, 16, 27, 41, \ldots \]

Exam #2

1.) A die is rolled ten times. What is the probability that it comes up 6 at least twice?

2.) A box contains eight red apples, six green apples and five plaid apples.
   a.) You reach into the box and pull out an apple at random. What is the probability the apple is plaid?
   b.) You reach into the box and pull out an apple at random. You then pull out another apple. What is the probability that both apples are green?
   c.) You reach into the box and pull out an apple at random. You then pull out another apple. What is the probability that both apples are the same color?
   d.) You reach into the box and pull out an apple at random. After putting it back, you then pull out another apple. What is the probability that both apples are the same color?

3.) A die is rolled until the first time it comes up with a 4. What is the average number of tosses required?

4.) Hank and Ted lost their coin. They are now using a die. If the die comes up 1 or 2, Hank takes a marker from Ted. Otherwise, Ted takes a marker from Hank. There are three markers and the game ends when one player has all three markers.
   a.) Identify the four states for this game.
   b.) Construct the transition matrix for this game.
   c.) Find the transition matrix for a two roll game.
5.) Team A has a probability of 2/3 of winning any individual game in a best of seven tournament against Team B.

a.) What is the probability that Team A loses the tournament?

b.) What is the probability that Team A wins the tournament in either four or five games?

6.) Twelve slips of paper are in a hat. They are numbered as follows. A piece of paper is chosen randomly from the hat.

1 2 2 2 5 6 6 7 7 11 12

a.) What is the expected value for the experiment?

b.) If a player wins a dollar amount equal to the square of the number (for example, he pulls a 5 and wins $25) he pulls from the hat, how much must the person running the game charge the player in order to break even?

7.) w, x, y and z are non-negative integers that are all less than or equal to 10. What is the probability that w + x + y + z ≤ 10?

Practice test

1.) You flip a coin 5 times. What is the probability that it comes up heads an even number of times?

2.) A class has 12 girls and 10 boys.

a.) If a five person committee is chosen at random, how many committees are possible?

b.) If a five person committee is chosen at random, how many committees are possible if the committee must have three girls and two boys?

c.) If a five person committee is chosen at random, how many committees are possible if the president and vice-president of the committee must be of different sexes?

d.) If a five person committee is chosen at random, what is the probability that the committee has an odd number of girls?

e.) If a five person committee is chosen at random, what is the probability that there is an odd number of boys?

3.) Ignoring years and February 29, what is the probability that, in a group of 12 people, no two people share the same birth date? What about in a group of 25 people?

4.) A baseball player gets a hit with a probability of 1/3 every time he comes to the plate.

a.) If he bats 10 times, what is the probability he gets exactly four hits?

b.) If he bats 10 times, what is the expected number of hits?

c.) What is the probability that he gets a hit in four straight at bats?
d.) If he is going to bat until he gets one hit, what is the average number of at bats it will take?

5.) Team A wins each game with a probability of 3/4. Suppose Team A and Team B are playing a best of five tournament.
   a.) What is the probability that Team A wins the tournament?
   b.) What is the expected number of games to be played in the tournament?
   c.) What is the probability that Team B wins in three games?
   d.) What is the probability that Team A loses at least one game?
   e.) What is the probability that Team A wins given that Team B wins the first game?

6.) Hank and Ted now have four markers. If the coin comes up heads, Hank takes a marker from Ted. Otherwise Ted takes a marker from Hank.
   a.) Identify the five states for this game.
   b.) Construct the transition matrix for this game.
   c.) Find the transition matrix for a two toss game.

7.) Three dice are tossed.
   a.) What is the probability that the product of the three dice is less than 71?
   b.) What is the probability that the product of the dice is 71?

8.) A coin is flipped until it comes up heads twice. What is the average number of tosses required?

Exam 3

1.) Obviously the following graph is planar. Count the numbers of edges, vertices and faces and verify that Euler's formula holds. (in case it is hard to see, everywhere lines intersect a vertex exists)

2.) Determine the following for the graph in #1.
   a.) What is the maximal cycle length?
   b.) Does an Eulerian trail exist? Why or why not?
   c.) How many edges are in the longest path?
   d.) Draw a spanning tree for the graph.

3.) Construct the following graphs.
   a.) $K_4$
   b.) $K_{3,4}$
   c.) a connected graph with degree sequence $6, 2, 2, 2, 2, 2, 2$.
   d.) a disconnected graph with degree sequence $3, 3, 3, 3, 2, 2, 2, 2$
4.) For each degree sequence below, determine whether or not a tree exists with the given sequence. If it does, construct it. If it doesn't, explain why not.

   a.) 5, 3, 1, 1, 1, 1, 1, 1
   b.) 5, 2, 1, 1, 1, 1, 1, 1
   c.) 8, 1, 1, 1, 1, 1, 1, 1
   d.) 7, 6, 5, 4, 3, 2, 1

5.) For each pair of graphs, define an isomorphism, if possible, between the two graphs. If it is not possible, give a reason why not.

   a.)

   b.)

6.) Prove that for every connected graph, G, if G has no cycles, then for every pair of vertices a and b in G, there is only one path from a to b in G.

Practice Test

1.) Which of the following graphs contain Eulerian trails or Eulerian circuits? Give reasons for your answers

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2.) For each of the graphs in #1, which are planar, which are not. Explain your answer.

3.) For each of the graphs in #1, is there a cycle of length 4?

4.) For each of the graphs in #1, find the length of a maximal cycle.

5.) For each of the graphs in #1, find the degree sequence.

6.) Fill in the blank and prove the result.
   $K_m$ contains an Eulerian circuit if and only if ____________________________.

7.) Fill in the blank and prove the result.
   $K_{m,n}$ contains an Eulerian circuit if and only if ____________________________.

8.) Prove that, in a simple, connected graph, two vertices must have the same degree.

9.) Prove that $K_{3,3}$ is not planar.

10.) Let $u, v$ be vertices in a tree, $T$, that do not share an edge. Prove that adding the edge $\{u, v\}$ to $T$ produces a graph, $T'$, that is not a tree.

11.) Let $T$ be a tree. Prove that removing any edge from $T$ produces a graph, $T'$, that is not connected.

Exam #4
1.) Construct the adjacency matrix for each of the following graphs.

![Graph A](image1)

2.) For the directed graph in #1, use the adjacency matrix to find how many walks of length two exist between each pair of vertices. Show your work and carefully list each pair of vertices along with the number of walks.

3.) Draw a tree that illustrates the wolf-goat-cabbage-traveler puzzle.

4.) Consider the arithmetic expression \((2 + 3 \times 6) / 4\).
   a.) Write the prefix notation for this expression.
   b.) Write the postfix notation for this expression.
   c.) Write the reverse Polish notation for this expression.
   d.) Draw a tree representing this arithmetic expression.

5.) Consider the graph below.
   a.) Find the Hamiltonian cycle (beginning and ending at A) with the lowest sum of weights.
Practice Test

1.) Construct the adjacency matrix for each of the following graphs.

2.) For the directed graph in #1, use the adjacency matrix to find how many walks of length four or less exist between each pair of vertices.

3.) A two-player game starts with 13 stones. Each player may remove 1, 2 or 3 stones in any turn. The player who takes the last stone loses. Draw a graph of the game and determine a winning strategy.

4.) Consider the arithmetic expression \(((2 + 3) \times 6) / 4 + 8\).
   a.) Write the prefix and postfix notation for this expression.
   b.) Write the reverse Polish notation for this expression.
   c.) Draw a tree representing this arithmetic expression.

5.) Consider the graph below.

b.) Find the Hamiltonian cycle (beginning and ending at A) with the highest sum of weights.
a.) Find the Hamiltonian cycle (beginning and ending at A) with the lowest sum of weights.

b.) Find the Hamiltonian cycle (beginning and ending at A) with the highest sum of weights.

Final Exam

1.) How many digits are there in $3456^{7890}$?

2.) How many numbers in {1, 2, 3, \ldots, 999, 1000} are divisible by 3 or 7?

3.) What is the probability that in a group of six people, two will have birthdays in the same month? (Assume all months are equally likely)

4.) What is the probability that three cards chosen randomly from a standard 52-card deck of cards consist of either three face cards (jack, queen and king are face cards) or three cards of the same suit (clubs, diamonds, spades or hearts)?

5.) If team A wins every game it plays with probability $3/4$, then what is the probability that team A wins a best-of-three series over team B?

6.) Suppose team A has a $1/4$ probability of winning each game in a best of seven series against team B. What is the expected number of games in the series?

7.) Prove that, in any graph, $G$, the number of nodes with odd degree is even.

8.) Prove that two connected, simple trees with the same degree sequence are not necessarily isomorphic.
9.) a.) Draw the graph $K_5$.
b.) Citing a theorem, determine whether or not $K_5$ admits an Eulerian circuit.
c.) If the answer to b.) is yes, draw the circuit. If the answer to b.) is no, show a Hamiltonian cycle in $K_5$.

10.) A game begins with 12 stones. Each player may remove one or two stones on each move with the one removing the last stone LOSING the game. Draw a graph that illustrates how this game can proceed and deduce a winning strategy, if possible, for the first player.

11.) Another game involved a queen on a chessboard (8 rows, 8 columns). The first player places the queen anywhere in the topmost row. The two players then alternate moves. A legal move is moving the queen any number of moves down or to the left. The queen may NOT be moved up or to the right. The winner is the player who places the queen on the bottom left square. Draw a graph that illustrates how this game can proceed and deduce a winning strategy, if possible, for the first player.